

Integrable Dissipative Nonlinear Second Order Differential Equations via Factorizations and Abel Equations

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We emphasize two connections, one well known and another less known, between the dissipative nonlinear second order differential equations and the Abel equations of the first kind having only cubic and quadratic terms. Then, employing an old integrability criterion due to Chiellini, we introduce the corresponding solvable dissipative equations. For illustration, we present the cases of some integrable dissipative Fisher, nonlinear pendulum, and Burgers-Huxley type equations which are obtained in this way.

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1.– Closed form solutions to a nonlinear ODE of the form

$$D_\zeta^2 u + g(u)D_\zeta u + h(u) = 0, \quad (1)$$

where $D_\zeta = \frac{d}{d\zeta}$ will be used alternatively in the following and g and h are general functions of $u(\zeta)$, can be obtained from the equivalence of (1) to the Abel equations of the second kind

$$\eta\eta_u + g(u)\eta + h(u) = 0, \quad (2)$$

as expressed by the following lemma.

Lemma 1: Solutions to a general second order ODE of type (1) may be obtained via the solutions to Abel's equation (2), and vice versa using the following relationship

$$\frac{du}{d\zeta} = \eta(u(\zeta)). \quad (3)$$

This equivalence can be found in the book of Polyanin and Zaitsev in the simpler case $h(u) = u$, which they call nonlinear oscillator equations [1].

Proof: To show the equivalence, one just need the simple chain rule

$$\frac{d^2 u}{d\zeta^2} = \frac{d\eta}{du} \frac{du}{d\zeta} \quad (4)$$

which turns (1) into

$$\frac{d\eta}{du} \frac{du}{d\zeta} + g(u) \frac{du}{d\zeta} + h(u) = 0, \quad (5)$$

which is (2).

Thus, the solutions of the several classes of known solvable Abel equations [2] are just the derivatives of the solutions of the second order nonlinear equations of the type (1). Hence, if one could solve (2) for η , then u could be found via inverting

$$\int^u \frac{1}{\eta} dr = \zeta - \zeta_0. \quad (6)$$

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Here, we refer to a powerful integrability result for first-kind Abel equations of the type (8) below, which is mentioned in the book of Kamke [3] as belonging to Chiellini [4]. Therefore, we will show that one does not need to find η from (2), but from a factorization technique instead. For other approaches to the integrability of Abel's equation and general overview we recommend [5].

Chiellini's integrability condition: If $g(u)$ and $h(u)$ are connected by

$$\frac{d}{du} \left(\frac{h(u)}{g(u)} \right) = kg(u), \quad (7)$$

for some constant k , then Abel's equation is integrable.

This important result has been almost forgotten in the literature and to the best of our knowledge it has been used only by Mak and Harko [6] in devising a method to get general solutions of the first-kind Abel equation from a particular solution.

Proof: Letting $\eta(u) = \frac{1}{y(u)}$, then equation (2) becomes

$$\frac{dy}{du} = h(u)y^3 + g(u)y^2, \quad (8)$$

which is an Abel equation of the first kind without linear and free terms. Furthermore, let $z = y \frac{h}{g}$ and take into account Chiellini's condition, then one gets

$$z_u = \frac{h}{g} y_u + kgy \quad (9)$$

and, therefore, the first-kind Abel equation (2) becomes

$$z_u = \frac{g^2}{h} (z^3 + z^2 + kz) \quad (10)$$

which is separable as follows

$$\int \frac{dz}{z(z^2 + z + k)} = \int^u \frac{g^2}{h} dr. \quad (11)$$

The right-hand-side of (11) can be written as $\frac{1}{k} \int \frac{d(\frac{h}{g})}{\frac{h}{g}}$. Therefore, one gets

$$\int \frac{dz}{z(z^2 + z + k)} = \frac{1}{k} \ln \left| \frac{h}{g} \right| + c. \quad (12)$$

2.- In this section, we will explain how we use the factorization method applied to (1) to obtain the solutions of (2). The factored form of (1) reads

$$[D_\zeta - \phi_2(u)][D_\zeta - \phi_1(u)]u(\zeta) = 0. \quad (13)$$

Expanding (13) and identifying terms, Rosu and Cornejo-Pérez [7] obtained the equation

$$D_\zeta^2 u - \left(\phi_1 + \phi_2 + \frac{d\phi_1}{du} u \right) D_\zeta u + \phi_1 \phi_2 u = 0, \quad (14)$$

which leads to the following conditions on the two factoring functions

$$\phi_1 \phi_2 = \frac{h(u)}{u}, \quad (15)$$

$$\phi_1 + \phi_2 = -\frac{d\phi_1}{du} u - g(u). \quad (16)$$

The solutions $\phi_1(u)$ and $\phi_2(u)$ of the system (15) are easily obtained by solving the quadratic equation

$$t^2 - St + P = 0, \quad (17)$$

where $S = -\frac{d\phi_1}{du}u - g(u)$, and $P = \frac{h(u)}{u}$, and $t^\pm = \phi_{1,2}$. By choosing $t^+ = \phi_1$, we obtain

$$\phi_1 \left(\phi_1 + \frac{d\phi_1}{du}u \right) + g\phi_1 + \frac{h}{u} = 0. \quad (18)$$

It is not a coincidence to notice that if

$$\eta(u) = u\phi_1(u), \quad (19)$$

then the equation for factors (18), is indeed Abel's equation (2). Therefore, rather than solving (2), η can be obtained from the factors of the ODE. Interestingly, the factorization method provides another argument for the two equivalences, (3) and (19). From the factorization (13) we have $[D_\zeta - \phi_1(u)]u(\zeta) = 0$, i.e., $D_\zeta u = \phi_1(u)u$. Thus, if we take $D_\zeta u = \eta$, then also $\phi_1(u)u = \eta$. In addition, the interpretation of the Abel solution as in (19) permits the formulation of another lemma as follows.

Lemma 2: For Chiellini-integrable ODEs, i.e., ODEs that have $g(u)$, and $h(u)$ connected via Chiellini's condition, the solution to Abel's equation (2) is given by

$$\eta(u) = c_k \frac{h(u)}{g(u)}, \quad (20)$$

where the constant c_k is given in terms of Chiellini's constant through

$$c_k = \frac{-1 \pm \sqrt{1 - 4k}}{2k}. \quad (21)$$

Proof: Let $\eta(u) = c_k \frac{h(u)}{g(u)}$, and substitute in (2), then one gets $kc_k^2 + c_k + 1 = 0$ with the roots given by (21).

Lemma 2 is very useful because one can employ it to find η from $g(u)$ and $h(u)$ as follows.

Theorem: For an integrable ODE of type (1),

i) if $g(u)$ is known, then

$$\eta_g(u) = c_k \left(c_0 + k \int^u g(r) dr \right) \quad (22)$$

or

ii) if $h(u)$ is known, then

$$\eta_h(u) = \pm c_k \sqrt{c_1 + 2k \int^u h(r) dr} \quad (23)$$

Proof: For both cases we will use Chiellini's condition together with lemma 2. For simplicity, we put $c_k = 1$.

i) if we have $g(u)$, then

$$h(u) = g(u) \left(c_0 + k \int^u g(r) dr \right) \quad (24)$$

by integrating (7).

ii) If we know $h(u)$, then we multiply equation (7) by $\frac{h}{g}$ to get

$$\frac{h}{g} \frac{d}{du} \left(\frac{h}{g} \right) = kh. \quad (25)$$

By integrating once with respect to u , we obtain

$$\frac{h^2}{g^2} = c_1 + 2k \int^u h(r) dr. \quad (26)$$

3.- We use now these results to obtain four integrable dissipative equations of type (1), either by starting with given g or h . We prefer to begin with the two cases of given h because usually the nonlinear equations are identified by their nonlinear term(s) and not so much by their dissipation coefficient. In these illustrative examples, we will take $c_k = 1$, which corresponds to $k = -2$.

A. Dissipative Fisher's equation

In this case, let $h(u) = u(1 - u)$. Using the theorem,

$$\eta_h(u) = \sqrt{c_1 - 2u^2 + \frac{4u^3}{3}}. \quad (27)$$

Then, one gets the following integrable dissipative Fisher's equation

$$u_{\zeta\zeta} + \frac{u(1-u)}{\sqrt{c_1 - 2u^2 + \frac{4u^3}{3}}} u_{\zeta} + u(1-u) = 0, \quad (28)$$

with closed form solution given by

$$\zeta - \zeta_0 = \frac{\sqrt{3}}{2} \int^u \frac{dr}{\sqrt{r^3 - \frac{3}{2}r^2 + c_2}}, \quad (29)$$

where $c_2 = \frac{3}{4}c_1$.

We notice that (28) can be also written in the convective form

$$u_{\zeta\zeta} + \mu(u)uu_{\zeta} + u(1-u) = 0, \quad (30)$$

where $\mu(u) = \frac{1-u}{\sqrt{c_1 - 2u^2 + \frac{4u^3}{3}}}$ can be interpreted as a tuning function of the convection.

In general, convective Fisher equations have been applied with interesting results in population dynamics [8], while the case of constant μ has been studied by Schönborn and collaborators [9].

The dissipative Fisher solution of (28) for $c_2 = \frac{1}{2}$ is displayed in Fig. 1, and it has the dark soliton profile $u(\zeta) = 1 - \frac{3}{2}\text{sech}^2\frac{\zeta}{2}$. On the other hand, when $c_2 = \frac{1}{4}$ the solution is the Jacobi elliptic sn function of the form $u(\zeta) = \frac{1-\sqrt{3}}{2} + \sqrt{3}\text{sn}^2\left(\frac{\zeta}{3^{1/4}}|2\right)$ as plotted in Fig. 2.

B. Dissipative nonlinear pendulum equation

For this case, $h(u) = \sin u$ and then $\eta_h(u)$ is given by

$$\eta_h(u) = \sqrt{c_3 + 4 \cos u}. \quad (31)$$

Therefore, one gets the dissipative integrable nonlinear pendulum equation

$$u_{\zeta\zeta} + \frac{\sin u}{\sqrt{c_3 + 4 \cos u}} u_{\zeta} + \sin u = 0, \quad (32)$$

with closed form solution in terms of the elliptic integral of the first kind $F(u|m)$

$$\zeta - \zeta_0 = \frac{\sqrt{2m}}{2} \int^u \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} = \frac{\sqrt{2m}}{2} F(u|m), \quad (33)$$

where $m = \frac{8}{c_3+4}$.

In the case of the dissipative nonlinear pendulum equation (32) for $m = 2$ the solution is the amplitude for Jacobi elliptic function $u(\zeta) = \text{am}(\zeta|2)$, where $\zeta = F(u|2)$ as displayed in Fig. 3. If $m = 1$, then $F(u|1) = \ln |\sec(u) + \tan(u)|$, and hence the solution to (32) is $u(\zeta) = \pm \arcsin(\tanh(\sqrt{2}\zeta))$, see Fig. 4. If $m = 8/9$, then $u(\zeta) = \text{am}(\frac{3\zeta}{2}|\frac{8}{9})$, see Fig. 5.

C. Burgers-Huxley type equation

An equation of this type can be obtained if we let $g(u) = \mu u$, which through (24) leads to $h(u) = \mu^2 u(\sqrt{c_0/\mu} - u)(\sqrt{c_0/\mu} + u)$. This is similar to a Huxley nonlinearity, although not for the typical range of the Huxley parameters. The solutions are obtained from

$$\zeta = -\frac{1}{\mu} \int^u \frac{dr}{r^2 - \frac{c_0}{\mu}}. \quad (34)$$

After inverting, this leads to three simple elementary solutions as follows

$$u(\zeta) = \begin{cases} \left(\frac{c_0}{\mu}\right)^{1/2} \tanh(\sqrt{\mu c_0} \zeta) & \text{if } \frac{c_0}{\mu} > 0 \\ \frac{1}{\mu \zeta} & \text{if } c_0 = 0 \\ \left(-\frac{c_0}{\mu}\right)^{1/2} \tan(\sqrt{-\mu c_0} \zeta) & \text{if } \frac{c_0}{\mu} < 0. \end{cases}$$

We do not plot these solutions because they are well-known elementary functions.

D. Generalized nonlinear pendulum equation with sine dissipation

If we take $g(u) = \sin u$, then (24) gives $h(u) = c_0 \sin u + \sin 2u$, and therefore this is a generalized nonlinear pendulum equation of the type

$$u_{\zeta\zeta} + \sin u u_{\zeta} + c_0 \sin u + \sin 2u = 0. \quad (35)$$

The solutions are obtained by inverting

$$\zeta = \int \frac{du}{c_0 + 2 \cos u}, \quad (36)$$

which leads to the following solutions

$$u(\zeta) = \begin{cases} 2 \arctan(2\zeta) & \text{if } c_0 = 2 \\ 2 \operatorname{arccot}(2\zeta) & \text{if } c_0 = -2 \\ 2 \arctan\left(\frac{(2+c_0) \tanh\left(\frac{1}{2}\sqrt{4-c_0^2}\zeta\right)}{\sqrt{4-c_0^2}}\right) & \text{if } |c_0| < 2 \\ 2 \arctan\left(\frac{(2+c_0) \tan\left(\frac{1}{2}\sqrt{4-c_0^2}\zeta\right)}{\sqrt{4-c_0^2}}\right) & \text{if } |c_0| > 2. \end{cases}$$

Plots of the last two cases for $c_0 = 1$ and $c_0 = 3$ are displayed in Figs. 6 and 7, respectively.

4.– In summary, we have shown that the connections between dissipative nonlinear second order differential equations and the integrable Abel equations can be very useful to extend the class of integrable dissipative nonlinear equations. The convective-like Fisher's equation and the dissipative nonlinear pendulum equation as well as the Burgers-Huxley type equation introduced here are such examples but many other equations can be generated in this way.

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- [1] A.D. Polyanin and V.F. Zaitsev, Handbook of exact solutions for ordinary differential equations, CRC Press (1995).
 - [2] E.S. Cheb-Terrab, A.D. Roche, Comp. Phys. Commun. 130, 204-231 (2000).
 - [3] E. Kamke, Differentialgleichungen: Lösungsmethoden und Lösungen, Chelsea, New York, (1959).
 - [4] A. Chiellini, Boll. Unione Mat. Italiana 10, 301-307 (1931).
 - [5] J.F. Cariñena, J. de Lucas, M.F. Rañada, Int. J. Theor. Phys. 50, 2114-2124 (2011).
 - [6] M.K. Mak, T. Harko, Comput. Math. Appl. 43, 91-94 (2002). M.K. Mak, H.W. Chan, T. Harko, *ibidem* 41, 1395-1401 (2001). T. Harko, M.K. Mak, *ibidem* 46, 849-853 (2003).
 - [7] H.C. Rosu, O. Cornejo-Pérez, Phys. Rev E **71**, 046607 (2005). O. Cornejo-Pérez, H.C. Rosu, Prog. Theor. Phys. **114**, 533-538 (2005).
 - [8] L. Giuggioli, V.M. Kenkre, Physica D 183, 245-259 (2003).
 - [9] O. Schönborn, S. Puri, R.C. Desai, Phys. Rev. E 49, 3480-3483 (1994).

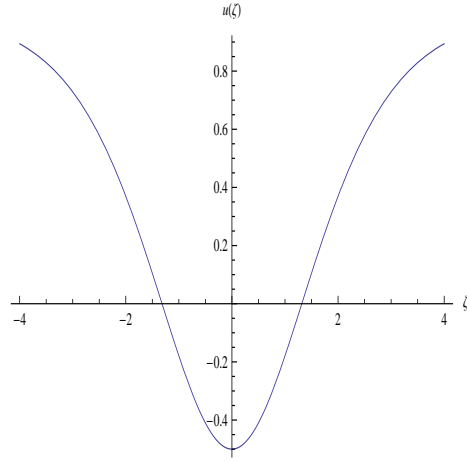


FIG. 1: (Color online) Solution with dark soliton profile, $u(\zeta) = 1 - \frac{3}{2}\text{sech}^2\frac{\zeta}{2}$, of the Abel-dissipative Fisher equation (28) for $c_1 = \frac{2}{3}$.

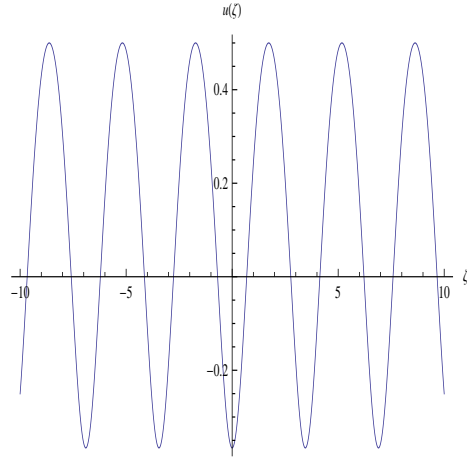


FIG. 2: (Color online) Elliptic function solution $u(\zeta) = \frac{1-\sqrt{3}}{2} + \sqrt{3}\text{sn}^2\left(\frac{\zeta}{3^{1/4}}|2\right)$ of the Abel-dissipative Fisher equation (28) for $c_1 = \frac{1}{3}$.

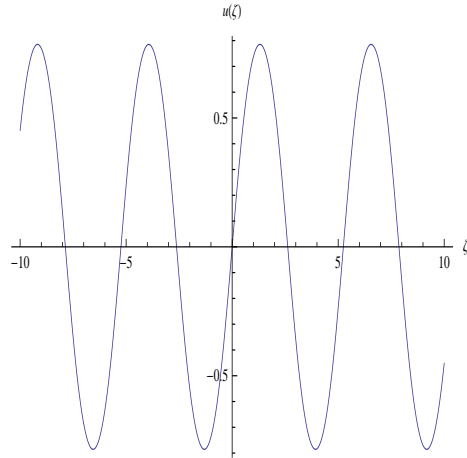


FIG. 3: (Color online) Amplitude for Jacobi elliptic function $u(\zeta) = \text{am}(\zeta|2)$ as solution of the dissipative pendulum equation (32) for $c_3 = 0$.

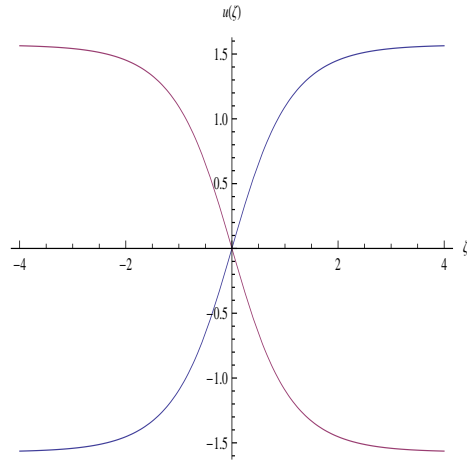


FIG. 4: (Color online) Solution $u(\zeta) = \pm \arcsin(\tanh(\sqrt{2}\zeta))$ of the dissipative pendulum equation (32) for $c_3 = 4$.

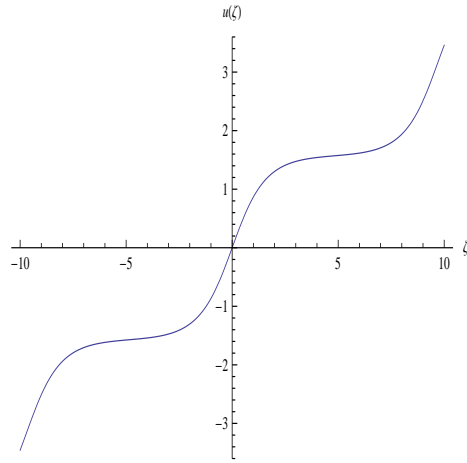


FIG. 5: (Color online) Solution $u(\zeta) = \text{am}(\frac{3\zeta}{2} | \frac{8}{9})$ of the dissipative pendulum equation (32) for $c_3 = 5$.

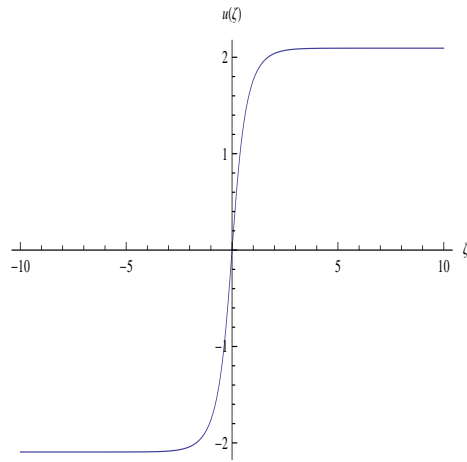


FIG. 6: (Color online) Solution $u(\zeta)$ of the generalized nonlinear pendulum equation (35) for $c_0 = 1$.

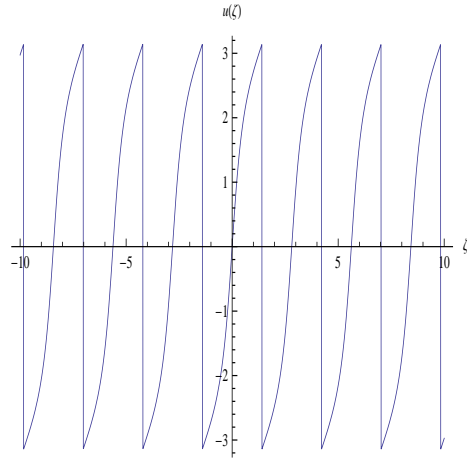


FIG. 7: (Color online) Solution $u(\zeta)$ of the generalized nonlinear pendulum equation (35) for $c_0 = 3$.